Mini-Course
Solving DSGE Models Using Perturbation Methods & Dynare

Brent Bundick
Federal Reserve Bank of Kansas City

April 2017

Goals of Mini-Course

Great Recession provides a number of new questions

Many questions are nonlinear in nature

1. Time-varying volatility and risk premia
2. Zero lower bound on nominal interest rates
3. Borrowing and lending in general equilbrium
4. Default decisions
5. Uncertainty about future fiscal policy
6. Fears of model miss-specification
7. Welfare costs of business cycles

Goal: Provide example of one type of nonlinear solution methods

Caveats of Mini-Course

Provide specific examples of methods using simple model

Focus is on implementation of methods

Very little discussion of theory due to time constraints

Please read and discuss the papers in reading list

Computational macroeconomics is art and science

1. Science of theory tells us what should work
2. Art of implementation helps us fix unforeseen issues

Many alternative methods and variations on implementation

Focus on solutions which have been successfully applied recently

Three Lessons For Using Numerical Methods

Lessons from my experience working with Susanto Basu

1. The economics, mathematics, & computer science must work together

    Computational economics is about developing code to solve models

    Your code will fail to solve many times

    Is it a problem with the economics, mathematics, or computer science?

    Numerical failures can teach us about the economics of the problem
    - See "Endogenous Volatility at the Zero Lower Bound"

Three Lessons For Using Numerical Methods

Lessons from my experience working with Troy Davig

2. Learn by doing and be persistent

   Be willing to teach yourself new methods and skills

   You will encounter setbacks but be persistent

   Balance between patience and persistence

Three Lessons For Using Numerical Methods

Lessons from my practical experience

3. Acknowledge the trade-off between your time & computational time

   I will never be the best mathematician or computer scientist

   As economists, we apply numerical methods to answer questions

   What is the opportunity cost of time spent optimizing your code?

   You could have great code, but a lousy written paper

   Rely on proven algorithms and routines as much as possible

Introduction of Specific Example

Focus on New-Keynesian sticky price model without capital

Shares features with models of Ireland (2003, 2010)

Household receives wages and lump-sum firm dividends

Standard preferences over streams of consumption and leisure

Household holds one-period nominal and real bonds

Fluctuations in household discount factor (demand shocks)

$$\beta_{t+1} = \beta \frac{a_{t+1}}{a_t}$$

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \sigma^a \varepsilon_t^a$$

Representative Household

Household maximizes lifetime utility from consumption and leisure

$$\max\ E_t \sum_{i=0}^{\infty} a_t \beta^i \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \chi \frac{N_t^{1+\eta}}{1+\eta} \right)$$

Household budget constraint

$$C_t + \frac{B_t}{P_t R_t} + \frac{B_t^R}{R_t^R} \leq \frac{W_t}{P_t} N_t + \frac{B_{t-1}}{P_t} + B_{t-1}^R + \frac{D_t}{P_t}$$

Household stochastic discount factor

$$M_{t+1} = \left(\beta \frac{a_{t+1}}{a_t} \right) \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma}$$

Representative Goods-Producing Firm

Firm i chooses $N_t(i)$ and $P_t(i)$ to maximize cash flows

$$\max\ E_t \left\{ \sum_{s=0}^{\infty} M_{t+s} \left(\frac{D_{t+s}(i)}{P_{t+s}} \right) \right\}$$

Definition of firm cash flows

$$\frac{D_t(i)}{P_t} = \left[\frac{P_t(i)}{P_t} \right]^{1-\theta} Y_t - \frac{W_t}{P_t} N_t(i) - \frac{\phi_P}{2} \left[\frac{P_t(i)}{\Pi P_{t-1}(i)} - 1 \right]^2 Y_t$$

Quadratic cost of changing nominal price $P_t(i)$

$$\frac{\phi_P}{2} \left[\frac{P_t(i)}{\Pi P_{t-1}(i)} - 1 \right]^2 Y_t$$

Cobb-Douglas production function subject to fixed costs

$$Y_t(i) = N_t(i) - \Phi$$

Aggregation & National Income Accounting

All users of final output assemble the final good Y_t using the range of varieties $Y_t(i)$ in a CES aggregator

$$Y_t = \left[\int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}}$$

Aggregate production function

$$Y_t = N_t - \Phi$$

National income accounting

$$Y_t = C_t + \frac{\phi_P}{2} \left(\frac{\Pi_t}{\Pi} - 1 \right)^2 Y_t$$

Model Summary

Model summarized by consumption Euler equations and NK Philips Curve

$$1 = \mathbb{E}_t \left\{ M_{t+1} \left(\frac{R_t}{\Pi_{t+1}} \right) \right\}$$

$$1 = \mathbb{E}_t \left\{ M_{t+1} R_t^R \right\}$$

$$\begin{aligned} \phi_P \left(\frac{\Pi_t}{\Pi} - 1 \right) \left(\frac{\Pi_t}{\Pi} \right) &= (1-\theta) + \theta \Xi_t \\ &+ \phi_P E_t \left\{ M_{t+1} \frac{Y_{t+1}}{Y_t} \left(\frac{\Pi_{t+1}}{\Pi} - 1 \right) \left(\frac{\Pi_{t+1}}{\Pi} \right) \right\} \end{aligned}$$

Model Summary II

Increase in a_t induces household to consume more & work less

Without zero lower bound, central bank can always stabilize economy

Economy experiences no fluctuations under flexible prices

Assume central bank follows Taylor (1993) monetary policy rule

$$\ln(R_t) = \ln(R) + \phi_\Pi \ln(\Pi_t/\Pi) + \phi_Y \ln(Y_t/Y)$$

Model Calibration and Solution

Calibrate model parameters to estimates of Ireland (2003, 2010)

How do we solve the model?

Solution is set of time-invariant policy functions which map exogenous and predetermined endogenous variables to each model variables

Our example model only has one state exogenous state: a_t

No endogenous state variables (no capital, price dispersion, debt, etc)

Solution is set of policy functions which satisfy all model equations

$C_t = C(a_t)$, $N_t = N(a_t)$, $\Pi_t = \Pi(a_t), \ldots$

Form of Solution

Our example model only has one state exogenous state: a_t

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \sigma^a \varepsilon_t^a$$

We can write policy functions in terms of a_t or $(a_{t-1}, \varepsilon_t^a)$

Use $(a_{t-1}, \varepsilon_t^a)$ form today

Can convert between two forms by substituting in a_t law of motion

Notation is special case of Swanson, Anderson, and Levine (2005)

Read Schmidt-Grohé and Uribe (2004) for alternative notation

Steady State

Solution is set of policy functions which satisfy all model equations

$$C_t = C\left(a_{t-1}, \varepsilon_t^a\right),\ N_t = N\left(a_{t-1}, \varepsilon_t^a\right),\ \Pi_t = \Pi\left(a_{t-1}, \varepsilon_t^a\right), \dots$$

We already know the solution at the deterministic steady state

$$a_{t-1} = a = 1, \quad \varepsilon_t^a = 0$$

$$C = C\left(a, 0\right),\ N = N\left(a, 0\right),\ \Pi = \Pi\left(a, 0\right), \dots$$

Finding steady state can be difficult numerically in practice

Can introduce constant in production function to fix steady state output Y

Type of Solution Method

Solution is set of policy functions which satisfy all model equations

$$C_t = C\left(a_{t-1}, \varepsilon_t^a\right),\ N_t = N\left(a_{t-1}, \varepsilon_t^a\right),\ \Pi_t = \Pi\left(a_{t-1}, \varepsilon_t^a\right), \ldots$$

What is the form of the rest of the policy function?

How to solve for its values?

Answer is determined by the economics of the problem

Two broad classes: Local & global approximations to the policy function

Local methods approximate function in neighborhood around the deterministic steady state

Global methods approximate function on a user-specified grid of points

Local Versus Global Solution Methods

Which type of solution method should be used?

Local methods can accurately & quickly approximate the policy functions of model around the steady state for many types of nonlinearities

Examples: Time-varying volatility, risk premia, and welfare

Local methods give approximation around single point in state space

Local methods cannot capture behavior far from steady state

Examples: Zero lower bound, models with default

Need approximation at many points in state space

Order of Approximation For Local Methods

Assume we can use local methods for our problem

Use Taylor-series approximation to policy functions of model around deterministic steady state

What order Taylor series to use?

Again, answer depends on economics of problem

1st-order linearization works for many macro applications

Need 2nd-order or higher to capture important nonlinearities

Order of Approximation Example

Do we need 2nd-order or higher to capture our nonlinearity?

Assume two different assets:

1. Risk-less bond which pays gross real interest rate R_t^R
2. Risky stock which has an expected rate of return $E_t R_{t+1}^S$

Earn same return to a 1st-order approximation (certainty equivalence)

Can determine *average* risk-premium $E_t R_{t+1}^S - R_t^R$ using 2nd-order

Need 3rd-order for risk premium that varies over time with state variables

Result emerges from how volatility of shocks enter the policy function

Solving For Policy Function - Introduce σ

How to solve for policy functions using Taylor series approximation?

$C_t = C\left(a_{t-1}, \varepsilon_t^a\right),\ N_t = N\left(a_{t-1}, \varepsilon_t^a\right),\ a_t = a\left(a_{t-1}, \varepsilon_t^a\right), \ldots$

Introduce identical scalar perturbation parameter $\sigma \in [0,1]$ into each policy function

$C_t = C\left(a_{t-1}, \varepsilon_t^a, \sigma\right),\ N_t = N\left(a_{t-1}, \varepsilon_t^a, \sigma\right),\ a_t = a\left(a_{t-1}, \varepsilon_t^a, \sigma\right), \ldots$

Substitute time-invariant policy functions into original nonlinear model

Scale $t+1$ shocks by σ

$$\frac{C_t}{C_{t+1}} = \frac{C\left(a_{t-1}, \varepsilon_t^a, \sigma\right)}{C\left(a_t, \varepsilon_{t+1}^a, \sigma\right)} = \frac{C\left(a_{t-1}, \varepsilon_t^a, \sigma\right)}{C\left(a\left(a_{t-1}, \varepsilon_t^a, \sigma\right), \sigma\varepsilon_{t+1}^a, \sigma\right)}$$

Solving For Policy Function - Know Steady State

$$\frac{C_t}{C_{t+1}} = \frac{C\left(a_{t-1}, \varepsilon_t^a, \sigma\right)}{C\left(a_t, \varepsilon_{t+1}^a, \sigma\right)} = \frac{C\left(a_{t-1}, \varepsilon_t^a, \sigma\right)}{C\left(a\left(a_{t-1}, \varepsilon_t^a, \sigma\right), \sigma\varepsilon_{t+1}^a, \sigma\right)}$$

Note $\sigma = 0$, $\varepsilon_t^a = 0$, $a_{t-1} = a$ corresponds to steady state

We already know the solution to each function at steady state

Note $\sigma = 1$ corresponds to the original model with uncertainty

Solving For Policy Function - System of Equations

Write model equations as system of equations

$$E_t F\left(x_{t-1}, x_t, x_{t+1}, \varepsilon_t, \sigma\varepsilon_{t+1}, \sigma\right) = 0$$

Solving for policy functions of the following form

$$x_t = b\left(x_{t-1}, \varepsilon_t, \sigma\right)$$

Substitute in policy functions into model equations

$$x_t = b\left(x_{t-1}, \varepsilon_t, \sigma\right)$$

$$x_{t+1} = b\left(x_t, \sigma\varepsilon_{t+1}, \sigma\right) = b\left(b\left(x_{t-1}, \varepsilon_t, \sigma\right), \sigma\varepsilon_{t+1}, \sigma\right)$$

Resulting system is function of state, policy functions, shocks, & σ

$$E_t\left(F \circ b\right)$$

First-Order Approximation of Policy Functions

Want first-order Taylor series approximation around point $(x_{t-1}, \varepsilon_t, \sigma) = (x, 0, 0)$

$$C_t = C\left(a,0,0\right) + C_a\left(a,0,0\right)\left(a_{t-1} - a\right) + C_\varepsilon\left(a,0,0\right)\varepsilon_t + C_\sigma\left(a,0,0\right)\sigma$$

Recall simple Taylor-series approximation for $F(x)$ at $x = a$

$$F(x) = F(a) + F_x(a)(x-a) + \frac{1}{2}F_{xx}(a)(x-a)^2 + \ldots$$

Take derivatives and evaluate at a known point (steady state)

Apply similar procedure to all model equations for each state variable

First-Order Approximation of Policy Functions

$$E_t\left(F\circ b\right)=0$$

1. Differentiate F with respect to all its arguments $(a_{t-1}, \varepsilon_t, \sigma, \sigma\varepsilon_{t+1})$ and evaluate them at the steady state $(a, 0, 0, 0)$. Note that these expressions will contain first-derivatives of unknown policy functions in b.
2. Form system of equations for each model equation as a function of first-derivatives of b.
3. Resulting system will be quadratic. Solve using standard methods as in Blanchard and Kahn (1985), Anderson and Moore (1985).
4. Solution to system gives the coefficients of each policy function as function of state, shocks, and σ

Properties of First-Order Approximation

$$C_t = C(a,0,0) + C_a(a,0,0)(a_{t-1} - a) + C_\varepsilon(a,0,0)\,\varepsilon_t + C_\sigma(a,0,0)\,\sigma$$

First derivative of policy function with respect to σ at $(a,0,0) = 0$

$$b_\sigma(x,0,0) = 0$$

σ does not appear in 1st-order approximation to policy functions

Volatility of shocks does not affect 1st-order approximation

Previous example: Volatility of stock return doesn't affect policy function

Higher-Order Approximations of Policy Functions

$$E_t\left(F \circ b\right) = 0$$

To compute $n \geq 2$ order approximations to policy functions

1. Differentiate F n-times with respect to all its arguments $(a_{t-1}, \varepsilon_t, \sigma, \sigma\varepsilon_{t+1})$ and evaluate them at the steady state $(a, 0, 0, 0)$. Note that these expressions will contain nth-order derivatives of unknown policy functions in b.
2. Form system of equations for each model equation as a function of nth-order derivatives of b.
3. Resulting system will be linear.
4. Solution to system gives the coefficients of each policy function as function of state, shocks, and σ

The procedure for an n-th order approximation is iterative:

Compute 1st order, 2nd order, ..., n-th order

Properties of Second-Order Approximation

$$
\begin{aligned}
C_t = \; & C\left(a,0,0\right)+C_a\left(a,0,0\right)\left(a_{t-1}-a\right)+C_\varepsilon\left(a,0,0\right)\varepsilon_t+C_\sigma\left(a,0,0\right)\sigma \\
& +\frac{1}{2}C_{aa}\left(a,0,0\right)\left(a_{t-1}-a\right)^2+\frac{1}{2}C_{\varepsilon\varepsilon}\left(a,0,0\right)\varepsilon_t^2+\frac{1}{2}C_{\sigma\sigma}\left(a,0,0\right)\sigma^2 \\
& +C_{a\varepsilon}\left(a,0,0\right)\left(a_{t-1}-a\right)\varepsilon_t+C_{a\sigma}\left(a,0,0\right)\left(a_{t-1}-a\right)\sigma \\
& +C_{\sigma\varepsilon}\left(a,0,0\right)\sigma\varepsilon_t
\end{aligned}
$$

Cross-derivative of policy function with respect to σ and any other state variable (a_{t-1}, ε_t) evaluated at $(a, 0, 0) = 0$

$$b_{\sigma x}\left(x,0,0\right)=b_{\sigma\varepsilon}\left(x,0,0\right)=0$$

2nd-derivative of policy function with respect to σ evaluated at $(a, 0, \sigma)$ is not zero

Recall $\sigma = 1$ corresponds to original model

Properties of Second-Order Approximation

$$
\begin{aligned}
C_t = {} & C(a,0,\sigma) + C_a(a,0,\sigma)(a_{t-1}-a) + C_\varepsilon(a,0,\sigma)\varepsilon_t + C_\sigma(a,0,\sigma)\sigma \\
& + \frac{1}{2}C_{aa}(a,0,\sigma)(a_{t-1}-a)^2 + \frac{1}{2}C_{\varepsilon\varepsilon}(a,0,\sigma)\varepsilon_t^2 + \frac{1}{2}C_{\sigma\sigma}(a,0,\sigma)\sigma^2 \\
& + C_{a\varepsilon}(a,0,\sigma)(a_{t-1}-a)\varepsilon_t + C_{a\sigma}(a,0,\sigma)(a_{t-1}-a)\sigma \\
& + C_{\sigma\varepsilon}(a,0,\sigma)\sigma\varepsilon_t
\end{aligned}
$$

$\sigma = 1$ implies that decision rules are shifted by a constant function of exogenous shock volatility

Volatility of shocks add an additional constant to policy functions

Previous example: Volatility of stock return implies constant risk premium

Ergodic mean of policy functions differs from deterministic steady state

Properties of Higher-Order Approximations

Cross-derivative of policy function with respect to σ any odd number of times and any other state variable (a_{t-1}, ε_t) evaluated at $(a, 0, 0) = 0$

Cross-derivative of policy function with respect to σ twice and any other state variable (a_{t-1}, ε_t) evaluated at $(a, 0, 0)$ is non-zero.

Previous example: Volatility of stock return implies time-varying risk premium even if underlying shock volatility is constant

Time-varying shock volatility enters as independent argument in 3rd-order

Implementation of Higher-Order Approximations

Method can be implemented in Matlab or Mathematica

Found lots of success with Perturbation AIM & Dynare

Always good to check final results with multiple implementations

Introduction to Dynare

Dynare is a pre-processor for Matlab or Octave

Write model file `BasicNK.mod` Run using command `dynare BasicNK`

Can run Matlab commands from inside model file

Introduction to Dynare

Dynare is a pre-processor for Matlab or Octave

Write model file `BasicNK.mod` Run using command `dynare BasicNK`

Can run Matlab commands from inside model file

Additional Concepts

Deterministic steady state

Place of rest for economy if no shocks in model

Average value for 1st-order solution

Stochastic steady state

Place of rest if economy is not hit by shocks, but agents expect them

Take higher-order solution, turn off shocks, iterate forward

Ergodic Mean

Average value if economy hit by shocks & agents expect them

Take higher-order solution, draw shocks, simulate & take average

Two Methods for Computing Impulse Responses

"No-Shock or Traditional" Impulse Response

Turn off shocks and iterate forward to stochastic steady state

Hit economy with single shock of interest & trace out response

Response centered around stochastic steady state

See `sss.m` & `irfsss.m` files in code

"Generalized Impulse Response"

Draw shocks randomly & generate numerous model simulations

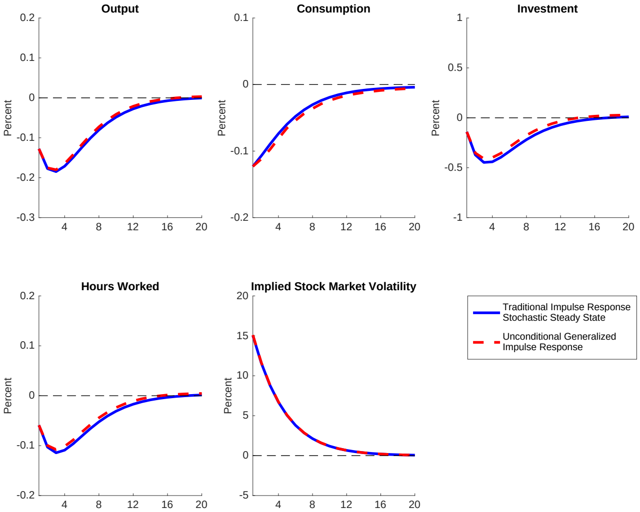
In particular period, hit economy with shock of interest

Average model simulations to see the average effect of given shock

Dynare's default method

Produce same results under 1st-order approximation

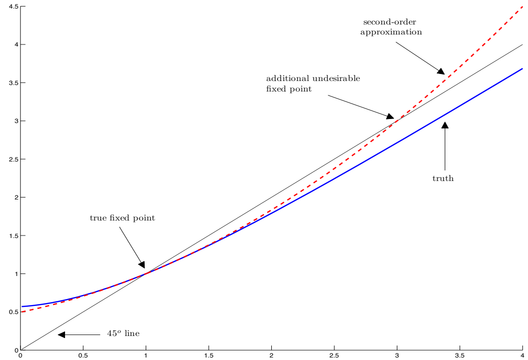
Alternative Impulse Response Construction



Generalized impulse response requires many simulations

Higher-Order Solutions Require Pruning in Simulations

Figure 1: Perturbation approximations and instability



Notes: This figure plots the function $f(x_{-1})$ described in section 1 and its second-order Taylor-series approximation.

See Den Haan and De Wind (2012) for detailed discussion